

Criticality of the Exponential Rate of Decay for the Largest Nearest Neighbor Link in Random Geometric Graphs

by

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Abstract

Let n points be placed independently in d -dimensional space according to the density $f(x) = A_d e^{-\lambda \|x\|^\alpha}$, $\lambda > 0$, $x \in \mathbb{R}^d$, $d \geq 2$. Let d_n be the longest edge length of the nearest neighbor graph on these points. We show that $(\lambda^{-1} \log n)^{1-1/\alpha} d_n - b_n$ converges weakly to the Gumbel distribution where $b_n \sim \frac{(d-1)}{\lambda\alpha} \log \log n$. We also prove the following strong law result for the normalized nearest neighbor distance $\tilde{d}_n := \frac{(\lambda^{-1} \log n)^{1-1/\alpha} d_n}{\log \log n}$.

$$\frac{d-1}{\alpha\lambda} \leq \liminf_{n \rightarrow \infty} \tilde{d}_n \leq \limsup_{n \rightarrow \infty} \tilde{d}_n \leq \frac{d}{\alpha\lambda},$$

almost surely. Thus, the exponential rate of decay $\alpha = 1$ is critical, in the sense that for $\alpha > 1$, $d_n \rightarrow 0$, whereas for $\alpha \leq 1$, $d_n \rightarrow \infty$ *a.s.* as $n \rightarrow \infty$.

May 30, 2009

AMS 1991 subject classifications:

Primary: 60D05, 60G70

Secondary: 05C05, 90C27

Keywords: Random geometric graphs, nearest neighbor graph, Poisson point process, largest nearest neighbor link, vertex degrees.

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1 Introduction and main results

In this paper we prove weak and strong law results for the largest nearest neighbor distance of points distributed according to the probability density function

$$f(x) = A_d e^{-\lambda \|x\|^\alpha}, \quad \lambda > 0, \alpha > 0, x \in \mathbb{R}^d, d \geq 2, \quad (1.1)$$

where $\|\cdot\|$ is the Euclidean (ℓ_2) norm on \mathbb{R}^d and

$$A_d = \frac{\alpha \lambda^{d/\alpha} \Gamma(d/2 + 1)}{d \pi^{d/2} \Gamma(d/\alpha)}. \quad (1.2)$$

If X has density given by (1.1), then $R = \|X\|$ has density,

$$f_R(r) = \frac{\alpha \lambda^{d/\alpha}}{\Gamma(d/\alpha)} r^{d-1} e^{-\lambda r^\alpha}, \quad 0 < r < \infty, d \geq 2. \quad (1.3)$$

The basic object of study will be the graphs G_n with vertex set $\mathcal{X}_n = \{X_1, X_2, \dots, X_n\}$, $n = 1, 2, \dots$, where the vertices are independently distributed according to f . Edges of G_n are formed by connecting each of the vertices in \mathcal{X}_n to its nearest neighbor. The longest edge of the graph G_n is denoted by d_n . We shall refer to G_n as the nearest neighbor graph (NNG) on \mathcal{X}_n and to d_n as the largest nearest neighbor distance (LNND). For any finite subset $\mathcal{X} \subset \mathbb{R}^d$, let $G(\mathcal{X}, r)$ denote the graph with vertex set \mathcal{X} and edges between all pairs of vertices that are at distances less than r . Thus, d_n is the minimum r_n required so that the graph $G(\mathcal{X}_n, r_n)$ has no isolated nodes.

The largest nearest neighbor link has been studied in the context of computational geometry (see Dette and Henze (1989) and Steele and Tierney (1986)) and has applications in statistics, computer science, biology and the physical sciences. For a detailed description

of Random Geometric Graphs, their properties and applications, we refer the reader to Penrose (2003) and references therein.

The asymptotic distribution of d_n was derived in Penrose (1997) assuming that f is uniform on the unit cube. It is shown that if the metric is assumed to be the toroidal, and if θ is the volume of the unit ball, then $n\theta d_n^d - b_n$ converge weakly to the Gumbel distribution, where $b_n \sim \log n$. Penrose (1998) showed that for normally distributed points ($\alpha = 2$), $\sqrt{(2 \log n)} d_n - b_n$ converge weakly to the Gumbel distribution, where $b_n \sim (d-1) \log \log n$. The above result is also shown to be true for the longest edge of the minimal spanning tree. The notation $a_n \sim b_n$ implies that a_n/b_n converges to one as $n \rightarrow \infty$. Hsing and Rootzen (2005) derive the asymptotic distribution for d_n in the case $d = 2$, for a large class of densities, including elliptically contoured distributions, distributions with independent Weibull-like marginals and distributions with parallel level curves (which includes the densities defined by (1.1)). Appel and Russo (1997) proved strong law results for d_n for graphs on uniform points in the d -dimensional unit cube. Penrose (1999) extended this to general densities having compact support Ω for which $\min_{x \in \Omega} f(x) > 0$.

Our aim in this paper is to show that when the tail of the density decays like an exponential or slower ($\alpha \leq 1$), d_n diverges, whereas for super exponential decay of the tail, $d_n \rightarrow 0$, a.s. as $n \rightarrow \infty$. Properties of the one dimensional exponential random geometric graphs have been studied in Gupta, Iyer and Manjunath (2005). In this case, spacings between the ordered nodes are independent and exponentially distributed. This allows for explicit computations of many characteristics for the graph and both strong and weak law results can be established.

It is often easier to study the graph G_n via the NNG P_n on the set $\mathcal{P}_n = \{X_1, X_2, \dots, X_{N_n}\}$, $n \geq 1$, where $\{N_n\}_{n \geq 1}$ is a sequence of Poisson random variables that are independent of the sequence $\{X_n\}_{n \geq 1}$ with $E[N_n] = n$. \mathcal{P}_n is an inhomogeneous Poisson point process with intensity function $n f(\cdot)$ (see Penrose (2003), Prop. 1.5). Note that the graphs G_n and P_n are coupled, since the first $\min(n, N_n)$ vertices of the two graphs are identical. We also assume that the random variables N_n are non-decreasing, so that $\mathcal{P}_1 \subset \mathcal{P}_2 \subset \mathcal{P}_3 \dots$.

Let $W_n(r_n)$ (respectively $W'_n(r_n)$) be the number of vertices of degree 0 (isolated nodes) in $G(\mathcal{X}_n, r_n)$ (respectively $G(\mathcal{P}_n, r_n)$). Let θ_d denote the volume of the d -dimensional unit ball in \mathbb{R}^d and let $Po(\lambda)$, denote a Poisson distribution with mean $\lambda > 0$. In what follows we will write $\log_2 n$ for $\log \log n$ and $\log_3 n$ for $\log \log \log n$ etc.

For any $\beta \in \mathbb{R}$, let $(r_n)_{n \geq 1}$ be a sequence of edge distances that satisfies

$$r_n(\lambda^{-1} \log n)^{1-1/\alpha} - \frac{(d-1)}{\lambda\alpha} \log_2 n + \frac{(d-1)}{2\lambda\alpha} \log_3 n \rightarrow \frac{\beta}{\lambda\alpha}, \quad (1.4)$$

as $n \rightarrow \infty$. We now state our main results.

Theorem 1.1 *Let $(r_n)_{n \geq 1}$ satisfy (1.4) as $n \rightarrow \infty$. Then,*

$$W_n(r_n) \rightarrow Po(e^{-\beta}/C_d) \quad (1.5)$$

in distribution, where

$$C_d = \frac{\alpha \theta_{d-1} (d-1)!}{2} \left(\frac{d-1}{2\pi} \right)^{\frac{d-1}{2}}. \quad (1.6)$$

An easy consequence of the above result is the following limiting distribution for d_n .

Theorem 1.2 *Let $f(\cdot)$ be the d -dimensional density defined as in (1.1). Let d_n be the largest nearest neighbor link of the graph G_n of n i.i.d. points $\mathcal{X}_n = \{X_1, X_2, \dots, X_n\}$ distributed according to f . Then,*

$$\lim_{n \rightarrow \infty} P[\lambda \alpha (\lambda^{-1} \log n)^{1-1/\alpha} d_n - (d-1) \log_2 n + \frac{(d-1)}{2} \log_3 n \leq \beta + \log(C_d)] \rightarrow \exp(-e^{-\beta}). \quad (1.7)$$

The above result for the case $\alpha = 2$, was derived in Penrose (1998). In dimension $d = 2$, Theorem 1.2 follows from Theorem 7, Hsing and Rootzen (2005) (see also Example 3). Their method is based on spatial blocking and uses a locally orthogonal coordinate system with respect to the level curves. We follow the approach in Penrose (1998) and use the Chen-Stein method.

Strong law results exist in the literature only for densities that do not vanish and whose support is bounded. Suppose $d \geq 2$, the density f is continuous, has support Ω , and that the boundary $\partial\Omega$ is a compact $(d-1)$ -dimensional C^2 submanifold of \mathbb{R}^d . Let $f_0 > 0$ be the essential infimum of f restricted to Ω , and $f_1 = \inf_{\partial\Omega} f$. Then (see Theorem 7.2, Penrose (2003)),

$$\lim_{n \rightarrow \infty} \frac{nd_n^d}{\log n} = \max \left\{ \frac{c_0}{f_0}, \frac{c_1}{f_1} \right\}, \quad \text{a.s.}$$

Thus, the asymptotic behavior of the LNND depends on the (reciprocal of the) infimum of the density, since it is in the vicinity of this infimum that points will be sparse and hence be farthest from each other. If f_0 or f_1 is zero, then the right hand side is infinite, implying that the scaling on the left is not the appropriate one. We now state a strong law result for

the largest nearest neighbor distance in our case.

Theorem 1.3 *Let d_n be the LNND of the NNG G_n defined on the collection \mathcal{X}_n of n points distributed independently and identically according to the density $f(\cdot)$ as defined in (1.1).*

Then, almost surely, for any $d \geq 2$,

$$\liminf_{n \rightarrow \infty} \frac{(\lambda^{-1} \log n)^{1-1/\alpha} d_n}{\log_2 n} \geq \frac{d-1}{\alpha\lambda}. \quad (1.8)$$

$$\limsup_{n \rightarrow \infty} \frac{(\lambda^{-1} \log n)^{1-1/\alpha} d_n}{\log_2 n} \leq \frac{d}{\alpha\lambda}. \quad (1.9)$$

2 Proofs and supporting results

For any $x \in \mathbb{R}^d$, let $B(x, r)$ denote the open ball of radius r centered at x . Let

$$I(x, r) := \int_{B(x, r)} f(y) dy. \quad (2.1)$$

For $\rho > 0$, define $I(\rho, r) = I(\rho e, r)$, where e is the d -dimensional unit vector $(1, 0, 0, \dots, 0)$.

Due to the radial symmetry of f , $I(x, r) = I(\|x\|, r)$. The following Lemma that provides a large ρ asymptotic for $I(\rho, r)$ will be crucial in subsequent calculations.

Lemma 2.1 *Let $d \geq 2$, and $(\rho_n)_{n \geq 1}$ and $(r_n)_{n \geq 1}$ be sequences of positive numbers satisfying*

$\rho_n \rightarrow \infty$, $r_n/\rho_n \rightarrow 0$, and $r_n^2 \rho_n^{\alpha-2} \rightarrow 0$, and $r_n \rho_n^{\alpha-1} \rightarrow \infty$. Then,

$$e^{-\lambda w_1(n)} \left(\Gamma \left(\frac{d+1}{2} \right) + E_n \right) H(n) \leq I(\rho_n, r_n) \leq e^{-\lambda w_2(n)} \Gamma \left(\frac{d+1}{2} \right) H(n), \quad (2.2)$$

where

$$w_1(n) = \begin{cases} \frac{\alpha}{2} r_n^2 (\rho_n^2 - 2r_n \rho_n)^{\frac{\alpha}{2}-1}, & 0 < \alpha \leq 2 \\ \frac{\alpha}{2} r_n^2 (\rho_n^2 + 2r_n \rho_n)^{\frac{\alpha}{2}-2} [(\alpha-1)\rho_n^2 + 2r_n \rho_n], & \alpha > 2, \end{cases} \quad (2.3)$$

$$w_2(n) = \begin{cases} \frac{\alpha(\alpha-2)}{2}(r_n\rho_n)^2(\rho_n - 2r_n\rho_n)^{\frac{\alpha}{2}-2}, & 0 < \alpha \leq 2 \\ 0, & \alpha > 2, \end{cases} \quad (2.4)$$

$$|E_n| \leq \frac{C_1}{r_n\rho_n^{\alpha-1}}, \quad (2.5)$$

$$H(n) = A_d\theta_{d-1}2^{\frac{d-1}{2}}r_n^d \exp(-\lambda(\rho_n^\alpha - \alpha r_n\rho_n^{\alpha-1}))(\lambda\alpha r_n\rho_n^{\alpha-1})^{-\frac{d+1}{2}}, \quad (2.6)$$

where A_d is as defined in (1.2), θ_{d-1} is the volume of the $(d-1)$ -dimensional unit ball, and C_1 is some constant. As $n \rightarrow \infty$, $E_n \rightarrow 0$, and $w_i(n) \rightarrow 0$, $i = 1, 2$.

Proof. In the definition of $I(\rho_n, r_n) = I(\rho_n e, r_n)$, set $y = (\rho_n + r_n t, r_n s)$, $t \in (-1, 1)$, $s \in \mathbb{R}^{d-1}$. This gives,

$$I(\rho_n, r_n) = A_d \int_{-1}^1 \int_{\|s\|^2 \leq (1-t^2), s \in \mathbb{R}^{d-1}} \exp\left(-\lambda((\rho_n + r_n t)^2 + (\|s\|r_n)^2)^{\frac{\alpha}{2}}\right) r_n^d ds dt. \quad (2.7)$$

Consider first the case $0 < \alpha \leq 2$. Using the Taylor's expansion we get,

$$\begin{aligned} ((\rho_n + r_n t)^2 + (\|s\|r_n)^2)^{\frac{\alpha}{2}} &= ((\rho_n^2 + 2r_n t \rho_n) + (t^2 + \|s\|^2)r_n^2)^{\frac{\alpha}{2}} \\ &= (\rho_n^2 + 2r_n \rho_n t)^{\frac{\alpha}{2}} + (r_n^2(t^2 + \|s\|^2))^{\frac{\alpha}{2}}(\rho_n^2 + 2r_n \rho_n t + \xi)^{\frac{\alpha}{2}-1} \\ &= (\rho_n^2 + 2r_n \rho_n t)^{\frac{\alpha}{2}} + h_1(n, s, t), \end{aligned} \quad (2.8)$$

where $h_1 = (r_n^2(t^2 + \|s\|^2))^{\frac{\alpha}{2}}(\rho_n^2 + 2r_n \rho_n t + \xi)^{\frac{\alpha}{2}-1}$, and $\xi \in (0, r_n^2(t^2 + \|s\|^2))$. Since $0 < \alpha \leq 2$, and $(t, s) \in B(0, 1)$, $0 \leq \xi \leq r_n^2$, and hence

$$\begin{aligned} 0 \leq h_1(n, s, t) &= (r_n^2(t^2 + \|s\|^2))^{\frac{\alpha}{2}}(\rho_n^2 + 2r_n \rho_n t + \xi)^{\frac{\alpha}{2}-1} \\ &\leq (r_n^2(t^2 + \|s\|^2))^{\frac{\alpha}{2}}(\rho_n^2 + 2r_n \rho_n t)^{\frac{\alpha}{2}-1} \leq w_1(n), \end{aligned}$$

where

$$0 \leq w_1(n) := \frac{\alpha}{2} r_n^2 (\rho_n^2 - 2r_n \rho_n)^{\frac{\alpha}{2}-1} = \frac{\alpha}{2} r_n^2 \rho_n^{\alpha-2} (1 - \frac{2r_n}{\rho_n})^{\frac{\alpha}{2}-1} \rightarrow 0, \quad (2.9)$$

since $r_n^2 \rho_n^{\alpha-2} \rightarrow 0$, and $r_n/\rho_n \rightarrow 0$ as $n \rightarrow \infty$. Again, from the Taylor's expansion applied

to $(\rho_n^2 + 2r_n \rho_n t)^{\alpha/2}$ in (2.8), we get

$$\begin{aligned} (\rho_n^2 + 2r_n \rho_n t)^{\frac{\alpha}{2}} &= \rho_n^\alpha + \alpha r_n t \rho_n^{\alpha-1} + \frac{\frac{\alpha}{2}(\frac{\alpha}{2} - 1)}{2} (2r_n t \rho_n)^2 (\rho_n^2 + \zeta)^{\frac{\alpha}{2}-2} \\ &= \rho_n^\alpha + \alpha r_n t \rho_n^{\alpha-1} + h_2(n, t), \end{aligned} \quad (2.10)$$

where $h_2(n, t) = \frac{\frac{\alpha}{2}(\frac{\alpha}{2}-1)}{2} (2r_n t \rho_n)^2 (\rho_n^2 + \zeta)^{\frac{\alpha}{2}-2}$, and $\zeta \in (\min(0, 2\rho_n r_n t), \max(0, 2\rho_n r_n t))$.

Since $0 < \alpha \leq 2$, and $-1 \leq t \leq 1$, we get

$$w_2(n) := \frac{\alpha(\alpha-2)}{2} r_n^2 \rho_n^{\alpha-2} \left(1 - 2\frac{r_n}{\rho_n}\right)^{\frac{\alpha}{2}-2} \leq h_2(n, t) \leq 0. \quad (2.11)$$

since $r_n^2 \rho_n^{\alpha-2} \rightarrow 0$, and $r_n/\rho_n \rightarrow 0$, it follows that $w_2(n) \rightarrow 0$ as $n \rightarrow \infty$. From (2.8)–(2.11)

we get

$$\rho_n^\alpha + 2\alpha r_n t \rho_n^{\alpha-1} + w_2 \leq ((\rho_n + r_n t)^2 + (\|s\| r_n)^2)^{\frac{\alpha}{2}} \leq \rho_n^\alpha + 2\alpha r_n t \rho_n^{\alpha-1} + w_1. \quad (2.12)$$

Using the above in (2.7), we get

$$A_d r_n^d e^{-\lambda w_1} G_n \leq I(\rho_n, r_n) \leq A_d r_n^d e^{-\lambda w_2} G_n, \quad (2.13)$$

where

$$G_n = \int_{-1}^1 \int_{\|s\|^2 \leq (1-t^2), s \in \mathbb{R}^{d-1}} \exp(-\lambda(\rho_n^\alpha + 2\alpha r_n t \rho_n^{\alpha-1})) \, ds \, dt, \quad (2.14)$$

and w_1, w_2 as defined in (2.9) and (2.11) respectively, converge to 0 as $n \rightarrow \infty$.

If $\alpha > 2$, then $h_2(n, t) \geq 0$, and we take w_1, w_2 to be the sums of the upper and lower

bounds of $h_1(n, s, t) + h_2(n, t)$ respectively. We then obtain (2.13) with $w_2(n) = 0$, and

$$\begin{aligned} w_1(n) &= \frac{\alpha}{2} r_n^2 (\rho_n^2 + 2r_n \rho_n)^{\frac{\alpha}{2}-1} + \frac{\alpha(\alpha-2)}{2} (r_n \rho_n)^2 (\rho_n^2 + 2r_n \rho_n)^{\frac{\alpha}{2}-2} \\ &= \frac{\alpha}{2} r_n^2 \rho_n^{\alpha-2} \left(1 + 2\frac{r_n}{\rho_n}\right)^{\frac{\alpha}{2}-1} + \frac{\alpha(\alpha-2)}{2} r_n^2 \rho_n^{\alpha-2} \left(1 + 2\frac{r_n}{\rho_n}\right)^{\frac{\alpha}{2}-2} \end{aligned} \quad (2.15)$$

which converges to zero by the conditions of the Lemma.

Now consider the integral in (2.14). First make the change of variable $u = t + 1$ and then set $v = \lambda \alpha r_n \rho_n^{\alpha-1} u$ to obtain

$$\begin{aligned}
G_n &= \theta_{d-1} e^{-\lambda \rho_n^\alpha} \int_{-1}^1 \exp(-\lambda \alpha r_n \rho_n^{\alpha-1} t) (1 - t^2)^{\frac{d-1}{2}} dt \\
&= \theta_{d-1} e^{-\lambda(\rho_n^\alpha - \alpha r_n \rho_n^{\alpha-1})} \int_0^2 \exp(-\lambda \alpha r_n \rho_n^{\alpha-1} u) u^{(d-1)/2} (2-u)^{(d-1)/2} du \\
&= \theta_{d-1} e^{-\lambda(\rho_n^\alpha - \alpha r_n \rho_n^{\alpha-1})} (\lambda \alpha r_n \rho_n^{\alpha-1})^{-\frac{d+1}{2}} 2^{\frac{d-1}{2}} K_n,
\end{aligned} \tag{2.16}$$

where,

$$K_n = \int_0^{2\lambda \alpha r_n \rho_n^{\alpha-1}} e^{-v} v^{\frac{d-1}{2}} \left(1 - \frac{v}{2\lambda \alpha r_n \rho_n^{\alpha-1}}\right)^{\frac{d-1}{2}} dv \leq \Gamma((d+1)/2). \tag{2.17}$$

We will show that as $r_n \rho_n^{\alpha-1} \rightarrow \infty$, the integral in (2.16) converges to $\Gamma((d+1)/2)$ and also estimate the error in this approximation. Write

$$E_n := K_n - \Gamma((d+1)/2) = A_n - B_n,$$

where,

$$\begin{aligned}
A_n &= \int_0^{2\lambda \alpha r_n \rho_n^{\alpha-1}} e^{-v} v^{(d-1)/2} \left[\left(1 - \frac{v}{2\lambda \alpha r_n \rho_n^{\alpha-1}}\right)^{(d-1)/2} - 1 \right] dv, \quad \text{and} \\
B_n &= \int_{2\lambda \alpha r_n \rho_n^{\alpha-1}}^\infty e^{-v} v^{(d-1)/2} dv. \\
|A_n| &\leq \sup_{0 \leq v \leq 2\lambda \alpha r_n \rho_n^{\alpha-1}} \left\{ e^{-v/2} \left| 1 - \left(1 - \frac{v}{2\lambda \alpha r_n \rho_n^{\alpha-1}}\right)^{(d-1)/2} \right| \right\} \int_0^\infty e^{-v/2} v^{(d-1)/2} dv.
\end{aligned}$$

Since $(1-x)^a \geq 1 - Cx$, $0 \leq x \leq 1$ with $C = 1_{\{0 < a \leq 1\}} + a 1_{\{a > 1\}}$, we get,

$$0 \leq 1 - \left(1 - \frac{v}{2\lambda \alpha r_n \rho_n^{\alpha-1}}\right)^{(d-1)/2} \leq \frac{Cv}{2\lambda \alpha r_n \rho_n^{\alpha-1}}, \quad 0 \leq v \leq 2\lambda \alpha r_n \rho_n^{\alpha-1},$$

$$\Rightarrow |A_n| \leq \frac{C}{2\lambda\alpha r_n \rho_n^{\alpha-1}} \sup_{0 \leq v < \infty} \left\{ v e^{-v/2} \right\} \int_0^\infty e^{-v/2} v^{(d-1)/2} dv = \left(\frac{C'}{r_n \rho_n^{\alpha-1}} \right),$$

where C' is some constant. Further,

$$|B_n| \leq e^{-\lambda\alpha r_n \rho_n^{\alpha-1}/2} \int_0^\infty e^{-v/2} v^{(d-1)/2} dv,$$

and hence decays exponentially fast in $r_n \rho_n^{\alpha-1}$. Putting the above two estimates in (2.16),

we get

$$|E_n| \leq \frac{C_1}{r_n \rho_n^{\alpha-1}} \rightarrow 0, \quad \text{as } n \rightarrow \infty. \quad (2.18)$$

The result now follows from (2.13), (2.16) and (2.18). \square

We first prove Theorem 1.1 for the number of isolated nodes $W'_n(r_n)$ in the Poisson graph $G(P_n, r_n)$. Towards this end, we first find an r_n for which $E[W'_n(r_n)]$ converges. From the Palm theory for Poisson processes (see (8.45), Penrose (2003)), we get

$$E[W'_n(r_n)] = n \int_{R^d} \exp(-nI(x, r_n)) f(x) dx.$$

Changing to Polar coordinates gives

$$E[W'_n(r_n)] = n \int_0^\infty \exp(-nI(s, r_n)) f_R(s) ds, \quad (2.19)$$

where f_R is defined in (1.3). Let $\rho_n(t)^\alpha := \frac{t+a_n}{\lambda}$, $t \geq -a_n$ where

$$a_n := [\log n + (d/\alpha - 1) \log_2 n - \log(\Gamma(d/\alpha))]. \quad (2.20)$$

The idea is to make a change of variable $t = \rho_n^{-1}(s)$ such that $n f_R(\rho_n(t)) \rho'_n(t)$ converges and then choose r_n so that the first factor in (2.19) also converges.

$$E[W'_n(r_n)] = \int_{-a_n}^\infty \exp(-nI(\rho_n(t), r_n)) g_n(t) dt, \quad (2.21)$$

where

$$\begin{aligned}
g_n(t) &:= n f_R(\rho_n(t)) \rho'_n(t) = \frac{n \lambda^{d/\alpha-1}}{\Gamma(d/\alpha)} \left(\frac{t+a_n}{\lambda} \right)^{\frac{d}{\alpha}-1} e^{-(t+a_n)} \\
&= \left(\frac{t+a_n}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \\
&= \left(\frac{t + \log n + (d/\alpha - 1) \log_2 n - \log(\Gamma(d/\alpha))}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \\
&\rightarrow e^{-t}, \quad \text{as } n \rightarrow \infty, \forall t \in \mathbb{R}.
\end{aligned} \tag{2.22}$$

Lemma 2.2 Suppose the sequence $\{r_n\}_{n \geq 1}$ satisfies (1.4). Let $t \in \mathbb{R}$, and set $\rho_n(t)^\alpha = \frac{t+a_n}{\lambda} 1_{\{t \geq -a_n\}}$, where a_n is as defined in (2.20). Then

$$\lim_{n \rightarrow \infty} n I(\rho_n, r_n) = C_d e^{\beta-t}, \tag{2.23}$$

where C_d is as defined in (1.6).

Proof. It is easy to verify that for each fixed $t \in \mathbb{R}$, $\rho_n = \rho_n(t), r_n$ satisfy the conditions of Lemma 2.1 and so we have

$$n I(\rho_n, r_n) \sim n A_d \theta_{d-1} 2^{\frac{d-1}{2}} \Gamma\left(\frac{d+1}{2}\right) r_n^d \exp(-\lambda(\rho_n^\alpha - \alpha r_n \rho_n^{\alpha-1})) (\lambda \alpha r_n \rho_n^{\alpha-1})^{-\frac{d+1}{2}}.$$

Substituting for $\lambda \rho_n^\alpha = t + \log n + (\frac{d}{\alpha} - 1) \log_2 n - \log(\Gamma(d/\alpha))$, we get

$$n I(\rho_n, r_n) \sim \frac{n A_d \theta_{d-1} 2^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2}) \Gamma(d/\alpha) e^{-t}}{n (\log n)^{\frac{d}{\alpha}-1}} r_n^d \exp(\lambda \alpha r_n \rho_n^{\alpha-1}) (\lambda \alpha r_n \rho_n^{\alpha-1})^{-\frac{d+1}{2}}. \tag{2.24}$$

From (1.4), we can write

$$r_n = \frac{d-1}{\lambda \alpha} \frac{\log_2 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} - \frac{d-1}{2 \lambda \alpha} \frac{\log_3 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} + \frac{\beta + o(1)}{\lambda \alpha (\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}}, \tag{2.25}$$

and hence

$$\begin{aligned}
\lambda \alpha r_n \rho_n^{\alpha-1} &= \left(\frac{(d-1) \log_2 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} - \frac{d-1}{2} \frac{\log_3 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} + \frac{\beta + o(1)}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} \right) \\
&\cdot \left(\frac{1}{\lambda} (t + \log n + (\frac{d}{\alpha} - 1) \log_2 n - \log(\Gamma(d/\alpha))) \right)^{\frac{\alpha-1}{\alpha}} \\
&= \left((d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta + o(1) \right) \\
&\cdot \left(1 + \frac{t}{\log n} + (\frac{d}{\alpha} - 1) \frac{\log_2 n}{\log n} - \frac{\log(\Gamma(d/\alpha))}{\log n} \right)^{\frac{\alpha-1}{\alpha}} \tag{2.26}
\end{aligned}$$

$$= (d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta + o(1). \tag{2.27}$$

Using (2.25) and (2.27) in (2.24), we get

$$\begin{aligned}
nI(\rho_n, r_n) &\sim \frac{A_d \theta_{d-1} 2^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2}) \Gamma(d/\alpha) e^{-t}}{(\log n)^{\frac{d}{\alpha}-1}} \\
&\cdot \left(\frac{d-1}{\lambda \alpha} \frac{\log_2 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} - \frac{d-1}{2\lambda \alpha} \frac{\log_3 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} + \frac{\beta}{\lambda \alpha (\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} \right)^d \\
&\cdot \left(\frac{\exp((d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta)}{((d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta)^{\frac{d+1}{2}}} \right) \\
&= A_d \theta_{d-1} 2^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2}) e^{\beta-t} \left(\frac{d-1}{\lambda \alpha} \frac{\log_2 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} \left(1 - \frac{\log_3 n}{2 \log_2 n} + \frac{\beta}{(d-1) \log_2 n} \right) \right)^d \\
&\cdot \frac{(\log n)^{d-1} (\log_2 n)^{-\frac{d-1}{2}}}{(\log n)^{\frac{d}{\alpha}-1}} \Gamma(d/\alpha) (d-1)^{-\frac{d+1}{2}} (\log_2 n)^{-\frac{d+1}{2}} \\
&\sim A_d \theta_{d-1} 2^{\frac{d-1}{2}} \Gamma(\frac{d+1}{2}) e^{\beta-t} \left(\frac{d-1}{\lambda \alpha} \frac{\log_2 n}{(\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} \right)^d \\
&\cdot (\log n)^{d-\frac{d}{\alpha}} (\log_2 n)^{-\frac{d-1}{2}} \Gamma(d/\alpha) (d-1)^{-\frac{d+1}{2}} (\log_2 n)^{-\frac{d+1}{2}} \rightarrow C_d e^{\beta-t}.
\end{aligned}$$

Lemma 2.3 *There exists a constant M depending on α, d and λ , such that the following*

inequalities hold for all large enough n .

1. Suppose $d/\alpha > 1$, and $\lambda r_n^\alpha - a_n \leq t \leq 0$, or $d/\alpha < 1$, and $-\frac{\log n}{\log_2 n} \leq t \leq 0$, then

$$g_n(t) \leq M e^{-t}.$$

2. For $d/\alpha < 1$, and $\lambda r_n^\alpha - a_n \leq t \leq -\frac{\log n}{\log_2 n}$, $g_n(t) \leq M \left(\frac{\log_2 n}{\log n} \right)^{d-\alpha} e^{-t}$.

Proof. Observe that for large n , $0.5 \log n \leq a_n \leq 2 \log n$, and $\lambda r_n^\alpha \geq \left(\frac{(d-1) \log_2 n}{2\alpha(\log n)^{1-1/\alpha}} \right)^\alpha$.

In the case when $d/\alpha > 1$, and $\lambda r_n^\alpha - a_n \leq t \leq 0$,

$$g_n(t) \leq \left(\frac{0 + a_n}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \leq 2^{\frac{d}{\alpha}-1} e^{-t}.$$

If $d/\alpha < 1$, and $-\frac{\log n}{\log_2 n} \leq t \leq 0$,

$$g_n(t) \leq \left(\frac{-\frac{\log n}{\log_2 n} + 0.5 \log n}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \leq 4^{1-\frac{d}{\alpha}} e^{-t}.$$

Finally, if $d/\alpha < 1$, and $\lambda r_n^\alpha - a_n \leq t \leq -\frac{\log n}{\log_2 n}$,

$$g_n(t) \leq \left(\frac{\lambda r_n^\alpha - a_n + a_n}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \leq \left(\frac{(d-1) \log_2 n}{2\alpha \log n} \right)^{d-\alpha} e^{-t}. \quad \square$$

We have the following proposition.

Proposition 2.4 *Let the sequence $\{r_n\}_{n \geq 1}$ satisfy (1.4). Then*

$$\lim_{n \rightarrow \infty} E[W'_n] = \frac{e^{-\beta}}{C_d}, \quad (2.28)$$

where C_d is as defined in (1.6).

Proof. From Lemma 2.2 and (2.22), for each $t \in \mathbb{R}$, we have

$$\lim_{n \rightarrow \infty} \exp(-nI(\rho_n(t), r_n)) g_n(t) = \exp(-C_d e^{\beta-t}) e^{-t}. \quad (2.29)$$

Suppose we can find integrable bounds for $\exp(-nI(\rho_n(t), r_n))g_n(t)$ that hold for all large n . Then from (2.21), (2.29) and the dominated convergence theorem, we have

$$\begin{aligned}\lim_{n \rightarrow \infty} E[W'_n(r_n)] &= \lim_{n \rightarrow \infty} \int_{-a_n}^{\infty} \exp(-nI(\rho_n(t), r_n))g_n(t)dt \\ &= \int_{-\infty}^{\infty} \exp(-C_d e^{\beta-t}) e^{-t} dt = \frac{e^{-\beta}}{C_d} \int_0^{\infty} e^{-y} dy = \frac{e^{-\beta}}{C_d}.\end{aligned}$$

We find integrable bounds for $\exp(-nI(\rho_n(t), r_n))g_n(t)$, by dividing the range of t into four parts.

1. First consider $t \geq 0$. For large n since $0.5 \log n < a_n < 2 \log n$, we have

$$g_n(t) \leq \begin{cases} \left(\frac{(t+2 \log n)}{\log n} \right)^{\frac{d}{\alpha}-1} e^{-t} \leq e^{-t} 2^{\frac{d}{\alpha}} \max(t, 1)^{\frac{d}{\alpha}-1}, & \frac{d}{\alpha} > 1, \\ \frac{e^{-t}}{2^{1-\frac{d}{\alpha}}}, & \frac{d}{\alpha} \leq 1. \end{cases} \quad (2.30)$$

By the above bound on $g_n(t)$, it follows that

$$\exp(-nI(\rho_n(t), r_n))g_n(t) \leq g_n(t), \quad (2.31)$$

is integrable over $(0, \infty)$.

2. Now consider the range $-\frac{\log n}{\log_2 n} \leq t \leq 0$. As $\lambda \rho_n(t)^\alpha = t + a_n$, from (2.26) we get

$$\begin{aligned}\lambda \alpha r_n \rho_n(t)^{\alpha-1} &= ((d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta + o(1)) \\ &\quad \cdot \left(1 + \frac{\alpha-1}{\alpha} \left(\frac{t + (\frac{d}{\alpha} - 1) \log_2 n - \log(\Gamma(d/\alpha))}{\log n} \right) (1 + \zeta_n(t))^{-\frac{1}{\alpha}} \right),\end{aligned}$$

where $|\zeta_n(t)| \leq |t + (d/\alpha - 1) \log_2 n - \log(\Gamma(d/\alpha))| (\log n)^{-1}$. Since, $-\frac{\log n}{\log_2 n} \leq t \leq 0$,

$\zeta_n(t) \rightarrow 0$, uniformly in $t \in \left(-\frac{\log n}{\log_2 n}, 0\right)$ as $n \rightarrow \infty$. Since $-1 \leq \frac{t \log_2 n}{\log n} \leq 0$, in the

above range of t , we can find constants c_1 and c_2 such that for n sufficiently large,

$$(d-1) \log_2 n - \frac{d-1}{2} \log_3 n - c_1 \leq \lambda \alpha r_n \rho_n(t)^{\alpha-1} \leq (d-1) \log_2 n - \frac{d-1}{2} \log_3 n + c_2. \quad (2.32)$$

Hence for all sufficiently large n we have

$$\exp(\lambda \alpha r_n \rho_n^{\alpha-1}) \geq \frac{(\log n)^{d-1}}{(\log_2 n)^{\frac{d-1}{2}}} e^{-c_1}. \quad (2.33)$$

From Lemma 2.1,

$$\begin{aligned} nI(\rho_n, r_n) &\geq nA_d \theta_{d-1} 2^{\frac{d-1}{2}} \left(\Gamma\left(\frac{d+1}{2}\right) + E_n \right) r_n^d e^{-\lambda w_1} \exp(-\lambda(\rho_n^\alpha - \alpha r_n \rho_n^{\alpha-1})) (\lambda \alpha r_n \rho_n^{\alpha-1})^{-\frac{d+1}{2}} \\ &= nA_d \theta_{d-1} 2^{\frac{d-1}{2}} \left(\Gamma\left(\frac{d+1}{2}\right) + E_n \right) r_n^d \frac{\Gamma(d/\alpha) e^{-t}}{n(\log n)^{d/\alpha-1}} e^{-\lambda w_1} \exp(\lambda \alpha r_n \rho_n^{\alpha-1}) (\lambda \alpha r_n \rho_n^{\alpha-1})^{-\frac{d+1}{2}}. \end{aligned}$$

Using (2.32) and (2.33) in above expression we get

$$\begin{aligned} nI(\rho_n, r_n) &\geq A_d \theta_{d-1} 2^{\frac{d-1}{2}} \left(\Gamma\left(\frac{d+1}{2}\right) + E_n \right) e^{-\lambda w_1(n)} \left(\frac{(d-1) \log_2 n}{\lambda \alpha (\lambda^{-1} \log n)^{1-1/\alpha}} + o(1) \right)^d \\ &\quad \cdot \frac{\Gamma(d/\alpha) e^{-t}}{(\log n)^{d/\alpha-1}} \frac{(\log n)^{d-1}}{(\log_2 n)^{\frac{d-1}{2}}} e^{-c_1} \left((d-1) \log_2 n - \frac{d-1}{2} \log_3 n + c_2 \right)^{-\frac{d+1}{2}} \\ &\geq C \left(\Gamma\left(\frac{d+1}{2}\right) + E_n \right) e^{-\lambda w_1(n)} e^{-t}. \end{aligned}$$

As in (2.32), for $-\frac{\log n}{\log_2 n} \leq t \leq 0$, it is easily verified that $r_n/\rho_n(t)$ and $r_n \rho_n(t)^{\alpha-2}$ converge uniformly to 0. It follows that $w_1(n)$ and E_n converge uniformly to 0. Hence, we can find a constant $c' > 0$, such that

$$nI(\rho_n, r_n) \geq c' e^{-t}.$$

From the above inequality and Lemma 2.3(1), there exists a constant c such that for all sufficiently large n , we have

$$\exp(-nI(\rho_n(t), r_n)) g_n(t) \leq c \exp(-c' e^{-t}) e^{-t}, \quad \left(-\frac{\log n}{\log_2 n}\right) \leq t \leq 0. \quad (2.34)$$

This upper bound is integrable over $t \in (-\infty, 0)$.

3. Next, consider the range $\lambda r_n^\alpha - a_n \leq t \leq -\frac{\log n}{\log_2 n}$. From the first inequality we have

$r_n \leq \rho_n(t)$, and hence

$$I(\rho_n(t), r_n) = \int_{B(\rho_n(t)e, r_n)} A_d e^{-\lambda \|x\|^\alpha} dx \quad (2.35)$$

$$> \int_{B(\rho_n(t)e, r_n), \|x\| \leq \rho_n(t)} A_d e^{-\lambda \|x\|^\alpha} dx \quad (2.36)$$

$$\geq A_d e^{-\lambda \rho_n(t)^\alpha} |B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))|, \quad (2.37)$$

where $|\cdot|$ denotes the volume and $e = (1, 0, \dots, 0) \in \mathbb{R}^d$. Inscribe a sphere of diameter r_n inside $B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))$ (see Figure 1). Hence,

$$|B(\rho_n(t)e, r_n) \cap B(0, \rho_n(t))| \geq \frac{\theta_d r_n^d}{2^d}. \quad (2.38)$$

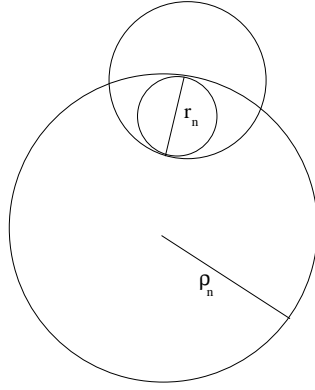


Figure 1

From (2.37) and (2.38), we have

$$\begin{aligned} I(\rho_n(t), r_n) &\geq c'' e^{-\lambda \rho_n(t)^\alpha} r_n^d \\ &= c'' \exp(-(t + \log n + (d/\alpha - 1) \log_2 n - \log(\Gamma(d/\alpha)))) \\ &\quad \cdot \left(\frac{(d-1) \log_2 n - \frac{d-1}{2} \log_3 n + \beta + o(1)}{\lambda \alpha (\lambda^{-1} \log n)^{1-\frac{1}{\alpha}}} \right)^d \\ &= \frac{c''' e^{-t}}{n (\log n)^{\frac{d}{\alpha}-1}} \frac{(\log_2 n)^d}{(\log n)^{d-\frac{d}{\alpha}}} \left(1 - \frac{\log_3 n}{2 \log_2 n} + \frac{\beta + o(1)}{(d-1) \log_2 n} \right)^d \end{aligned}$$

$$\geq c^* n^{-1} (\log n)^{1-d} (\log_2 n)^d e^{-t} = q_n e^{-t}, \quad (2.39)$$

where $q_n = c^* (\log n)^{1-d} (\log_2 n)^d n^{-1}$. From Lemma 2.3 and (2.39) we get,

$$\begin{aligned} \int_{\lambda r_n^\alpha - a_n}^{-\frac{\log n}{\log_2 n}} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt &\leq \begin{cases} M \int_{\lambda r_n^\alpha - a_n}^{-\frac{\log n}{\log_2 n}} \exp(-nq_n e^{-t}) e^{-t} dt, & \frac{d}{\alpha} \geq 1; \\ M \left(\frac{\log_2 n}{\log n} \right)^{d-\alpha} \int_{\lambda r_n^\alpha - a_n}^{-\frac{\log n}{\log_2 n}} \exp(-nq_n e^{-t}) e^{-t} dt, & \frac{d}{\alpha} < 1. \end{cases} \\ &\leq \begin{cases} M \int_{\exp(\frac{\log n}{\log_2 n})}^{\exp(a_n - \lambda r_n^\alpha)} e^{-nq_n y} dy, & \frac{d}{\alpha} \geq 1; \\ M \left(\frac{\log_2 n}{\log n} \right)^{d-\alpha} \int_{\exp(\frac{\log n}{\log_2 n})}^{\exp(a_n - \lambda r_n^\alpha)} e^{-nq_n y} dy, & \frac{d}{\alpha} < 1. \end{cases} \\ &\leq \begin{cases} \frac{M}{nq_n} e^{-nq_n e^{\frac{\log n}{\log_2 n}}}, & \frac{d}{\alpha} \geq 1; \\ \frac{M}{nq_n} \left(\frac{\log_2 n}{\log n} \right)^{d-\alpha} e^{-nq_n e^{\frac{\log n}{\log_2 n}}}, & \frac{d}{\alpha} < 1. \end{cases} \end{aligned} \quad (2.40)$$

$$\begin{aligned} \frac{M}{nq_n} e^{-nq_n e^{\frac{\log n}{\log_2 n}}} &= \frac{M}{nq_n} \exp \left(-n^{1+\frac{1}{\log_2 n}} q_n \right) \\ &= C \frac{(\log n)^{d-1}}{(\log_2 n)^d} \exp \left(-c^* n^{\frac{1}{\log_2 n}} (\log n)^{1-d} (\log_2 n)^d \right). \end{aligned} \quad (2.41)$$

Consider the exponent $c^* n^{\frac{1}{\log_2 n}} (\log n)^{1-d} (\log_2 n)^d$. Taking logarithms, we get

$$\log(c^*) + \frac{\log n}{\log_2 n} + (1-d) \log_2 n + d \log_3 n \geq \frac{\log n}{2 \log_2 n}.$$

Hence,

$$c^* n^{\frac{1}{\log_2 n}} (\log n)^{1-d} (\log_2 n)^d \geq e^{\frac{\log n}{2 \log_2 n}}. \quad (2.42)$$

Using (2.42) in (2.41), we get

$$\frac{M}{nq_n} e^{-nq_n e^{\frac{\log n}{\log_2 n}}} \leq C \left(\frac{\log n}{\log_2 n} \right)^{d-1} \frac{1}{\log_2 n} \exp \left(-e^{\frac{1}{2} \frac{\log n}{\log_2 n}} \right) \rightarrow 0, \quad (2.43)$$

since the exponent is decaying exponentially fast in $(\log n / \log_2 n)$. Using the inequality

from (2.43) in (2.40) for the case $d/\alpha < 1$, we get

$$\frac{M}{nq_n} \left(\frac{\log_2 n}{\log n} \right)^{d-\alpha} e^{-nq_n e^{\frac{\log n}{\log_2 n}}} \leq \frac{C(\log n)^{\alpha-1}}{(\log_2 n)^\alpha} \exp \left(-e^{\frac{\log n}{2\log_2 n}} \right), \quad (2.44)$$

which converges to 0, as $n \rightarrow \infty$, by the same argument as above. From (2.40), (2.43)

and (2.44) we have

$$\int_{\lambda r_n^\alpha - a_n}^{-\frac{\log n}{\log_2 n}} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt \rightarrow 0. \quad (2.45)$$

4. Finally, consider the case $-a_n \leq t \leq \lambda r_n^\alpha - a_n$. The second inequality implies that

$r_n \geq \rho_n(t)$. Hence for large n we have,

$$\begin{aligned} nI(\rho_n(t), r_n) &= n \int_{B(\rho_n(t), r_n)} A_d e^{-\lambda \|x\|^\alpha} dx \\ &> n \int_{B(r_n, r_n)} A_d e^{-\lambda \|x\|^\alpha} dx \geq c_1 n e^{-\lambda (2r_n)^\alpha} r_n^d. \end{aligned} \quad (2.46)$$

For large n from (2.25), we have

$$\frac{(d-1)\log_2 n}{2\lambda^{\frac{1}{\alpha}} \alpha (\log n)^{1-\frac{1}{\alpha}}} \leq r_n \leq \frac{2(d-1)\log_2 n}{\lambda^{\frac{1}{\alpha}} \alpha (\log n)^{1-\frac{1}{\alpha}}}. \quad (2.47)$$

Fix $0 < \epsilon_1, \epsilon_2 < 1$, such that $\epsilon = \epsilon_1 + \epsilon_2 < 1$. Substituting from (2.47) in (2.46), we

get, for large n ,

$$\begin{aligned} nI(\rho_n(t), r_n) &\geq c_2 n e^{-c_3 \frac{(\log_2 n)^\alpha}{(\log n)^{\alpha-1}}} \frac{(\log_2 n)^d}{(\log n)^{d-\frac{d}{\alpha}}} \\ &\geq c_2 n^{1-\epsilon_1} e^{-c_3 \left(\frac{\log_2 n}{\log n} \right)^\alpha \log n} \\ &= c_2 n^{1-\epsilon_1-c_3 \left(\frac{\log_2 n}{\log n} \right)^\alpha} \geq c_2 n^{1-\epsilon_1-\epsilon_2} = c_2 n^{1-\epsilon}. \end{aligned} \quad (2.48)$$

From (2.22), (2.48) and the fact that for large n , $a_n < 2 \log n$, we get

$$\int_{-a_n}^{\lambda r_n^\alpha - a_n} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt \leq \frac{e^{-c_2 n^{1-\epsilon}}}{(\log n)^{\frac{d}{\alpha}-1}} \int_{-a_n}^{\lambda r_n^\alpha - a_n} (t + a_n)^{\frac{d}{\alpha}-1} e^{-t} dt$$

$$\begin{aligned}
&\leq \frac{e^{a_n} e^{-c_2 n^{1-\epsilon}}}{(\log n)^{\frac{d}{\alpha}-1}} \int_0^\infty u^{\frac{d}{\alpha}-1} e^{-u} du \\
&\leq \frac{c n^2 e^{-c_2 n^{1-\epsilon}}}{(\log n)^{\frac{d}{\alpha}-1}} \rightarrow 0.
\end{aligned} \tag{2.49}$$

This completes the proof of Proposition 2.4. \square

Theorem 2.5 *Let $\alpha \in \mathbb{R}$ and let r_n be as defined in (1.4). Then,*

$$W'_{0,n}(r_n) \xrightarrow{\mathcal{D}} Po(e^{-\beta}/C_d),$$

where C_d is as defined in (1.6) and $Po(e^{-\beta}/C_d)$ is the Poisson random variable with mean $e^{-\beta}/C_d$.

Proof. From Theorem 6.7, Penrose (2003), we have $d_{TV}(W'_{0,n}(r_n), Po(E(W'_{0,n}(r_n))))$ is bounded by a constant times $J_1(n) + J_2(n)$ where $J_1(n)$ and $J_2(n)$ are defined as follows.

$$J_1(n) = n^2 \int_{\mathbb{R}^d} \exp(-nI(x, r_n)) f(x) dx \int_{B(x, 3r_n)} \exp(-nI(y, r_n)) f(y) dy, \tag{2.50}$$

and

$$J_2(n) = n^2 \int_{\mathbb{R}^d} f(x) dx \int_{B(x, 3r_n) \setminus B(x, r_n)} \exp(-nI^{(2)}(x, y, r_n)) f(y) dy, \tag{2.51}$$

where $I^{(2)}(x, y, r) = \int_{B(x, r) \cup B(y, r)} f(z) dz$. Theorem 2.5 follows from Proposition 2.4 if we show that $J_i(n) \rightarrow 0$, as $n \rightarrow \infty$, $i = 1, 2$. We first analyze J_1 . Let $\rho_n(t)$, $g_n(t)$ be as defined in Lemma 2.2 and (2.22) respectively.

$$\begin{aligned}
J_1(n) &= n^2 \int_{-a_n}^\infty \exp(-nI(\rho_n(t), r_n)) g_n(t) dt \int_{B(\rho_n(t)e, 3r_n)} \exp(-nI(y, r_n)) f(y) dy \\
&= J_{11}(n) + J_{12}(n),
\end{aligned}$$

where $J_{11}(n)$, and $J_{12}(n)$ are defined as follows:

$$\begin{aligned} J_{11}(n) &= \int_{-a_n}^{-\frac{\log n}{\log 2}} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt \int_{B(\rho_n(t)e, 3r_n)} \exp(-nI(y, r_n)) n f(y) dy, \\ J_{12}(n) &= \int_{-\frac{\log n}{\log 2}}^{\infty} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt \int_{B(\rho_n(t)e, 3r_n)} \exp(-nI(y, r_n)) n f(y) dy. \end{aligned}$$

First we will show that $J_{11}(n) \rightarrow 0$. From Proposition 2.4, the inner integral in J_{11} ,

$$\begin{aligned} \int_{B(\rho_n(t), 3r_n)} \exp(-nI(y, r_n)) n f(y) dy &\leq \int_{-a_n}^{\infty} \exp(-nI(\rho_n(t'), r_n)) g_n(t') dt' \\ &= E(W'_n(r_n)) \rightarrow \frac{e^{-\beta}}{C_d}, \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, for any $\epsilon > 0$, and all large n , we have

$$J_{11}(n) \leq (1 + \epsilon) \frac{e^{-\beta}}{C_d} \int_{-a_n}^{-\frac{\log n}{\log 2}} \exp(-nI(\rho_n(t), r_n)) g_n(t) dt. \quad (2.52)$$

It follows from (2.45), (2.49) that $J_{11}(n) \rightarrow 0$. Next we will show that $J_{12}(n) \rightarrow 0$ as $n \rightarrow \infty$.

Define $B_n(t) = \{t' : \rho_n(t) - 3r_n \leq \rho_n(t') \leq \rho_n(t) + 3r_n\}$. The inner integral in $J_{12}(n)$,

$$\begin{aligned} &\int_{B(\rho_n(t)e, 3r_n)} \exp(-nI(\rho_n(t'), r_n)) g_n(t') dt' \\ &\leq \left(2 \sin^{-1} \left(\frac{3r_n}{\rho_n(t)} \right) \right)^{d-1} \int_{B_n(t)} \exp(-nI(\rho_n(t'), r_n)) g_n(t') dt' \\ &\leq \left(2 \sin^{-1} \left(\frac{3r_n}{\rho_n(t)} \right) \right)^{d-1} \int_{-a_n}^{\infty} \exp(-nI(\rho_n(t'), r_n)) g_n(t') dt' \\ &\leq 2^{d-1} (1 + \epsilon) \frac{e^{-\beta}}{C_d} \left(\sin^{-1} \left(\frac{3r_n}{\rho_n(t)} \right) \right)^{d-1} \leq C \left(\frac{\log_2 n}{\log n} \right)^{d-1}, \end{aligned}$$

since for all large n , and $t \in (-\frac{\log n}{\log 2}, \infty)$, we can find constants c, c' and $\epsilon > 0$ such that

$0 \leq \frac{3r_n}{\rho_n(t)} \leq c \frac{\log_2 n}{\log n} \rightarrow 0$, and $\sin^{-1}(x) \leq c'x$, for all $x \in [0, \epsilon]$. Thus the inner integral in J_{12}

converges uniformly to 0, as $n \rightarrow \infty$. Hence J_{12} converges to 0 from the last statement and

the fact that the expressions in (2.31), (2.34) are integrable.

We now show that J_2 as defined in (2.51) converges to 0. Write

$$J_2(n) = J_{21}(n) + J_{22}(n) + J_{23}(n), \quad (2.53)$$

where

$$J_{2k}(n) = n^2 \int_{\mathbb{R}^d} f(x) dx \int_{A_k(n)} \exp(-nI^{(2)}(x, y, r_n)) f(y) dy, \quad k = 1, 2, 3,$$

with $A_1(n) = \{2r_n \leq \|x - y\| \leq 3r_n\}$, $A_2(n) = \{r_n \leq \|x - y\| \leq 2r_n, \|x\| \leq \|y\|\}$, and $A_3(n) = \{r_n \leq \|x - y\| \leq 2r_n, \|y\| \leq \|x\|\}$. Since on $A_1(n)$, $I^{(2)}(x, y, r_n) = I(x, r_n) + I(y, r_n)$, we get,

$$\begin{aligned} J_{21}(n) &= n^2 \int_{\mathbb{R}^d} \exp(-nI(x, r_n)) f(x) dx \int_{\{y: 2r_n \leq \|x-y\| \leq 3r_n\}} \exp(-nI(y, r_n)) f(y) dy, \\ &\leq n^2 \int_{\mathbb{R}^d} \exp(-nI(x, r_n)) f(x) dx \int_{B(x, 3r_n)} \exp(-nI(y, r_n)) f(y) dy = J_1(n), \end{aligned}$$

which has already been shown to converge to 0. Next we analyze $J_{22}(n)$ as $n \rightarrow \infty$. The proof for $J_{23}(n)$ is the same and so we omit it.

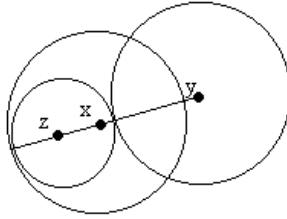


Figure 2

Let $B(z(x, y), \rho_1)$ be the ball with center $z = z(x, y)$ (see Figure 2) and radius $\rho_1 = \rho_1(x, y) \geq \frac{r_n}{2}$ inscribed inside $B(x, r_n) \setminus B(y, r_n)$. Then

$$I^{(2)}(x, y, r_n) \geq I(z(x, y), \rho_1) + I(y, r_n)$$

$$\begin{aligned}
&\geq I(z(x, y), r_n/2) + I(y, r_n) \\
&\geq I(x, r_n/2) + I(y, r_n),
\end{aligned}$$

where the last inequality follows since $\|z\| < \|x\|$. Thus,

$$\begin{aligned}
J_{22}(n) &\leq n^2 \int_{\mathbb{R}^d} \exp(-nI(x, r_n/2)) f(x) dx \int_{A_2(n)} \exp(-nI(y, r_n)) f(y) dy \\
&\leq n^2 \int_{\mathbb{R}^d} \exp(-nI(x, r_n/2)) f(x) dx \int_{B(x, 3r_n)} \exp(-nI(y, r_n)) f(y) dy \\
&= J_1^*(n) + J_2^*(n) + J_3^*(n),
\end{aligned}$$

where

$$J_i^* = \int_{D_i} \exp(-nI(\rho_n(t), r_n/2)) g_n(t) dt \int_{B(\rho_n(t), 3r_n)} \exp(-nI(y, r_n)) n f(y) dy, \quad i = 1, 2, 3, \quad (2.54)$$

where $D_1 = [-a_n, -\frac{\log n}{\log_2 n})$, $D_2 = [-\frac{\log n}{\log_2 n}, 0)$ and $D_3 = [0, \infty)$. The proof of $J_i^* \rightarrow 0$, as $n \rightarrow \infty$, for $i = 1, 3$ proceed exactly in the same manner as in the case of J_{11} and J_{12} by replacing r_n by $r_n/2$ while estimating the outer integrals. In the case of J_2^* , we proceed exactly as in the case of J_{12} to obtain

$$J_2^* \leq C \left(\frac{\log_2 n}{\log n} \right)^{d-1} \int_{D_2} \exp(-nI(\rho_n(t), r_n/2)) g_n(t) dt.$$

Estimating the integrand in the same way as in (2.34), with r_n replaced by $r_n/2$ and integrating, we get

$$J_2^* \leq C' \left(\frac{\log_2 n}{\log n} \right)^{d-1} \left(\frac{(\log n)^{\frac{d-1}{2}}}{(\log_2 n)^{\frac{d-1}{4}}} \right) \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

This completes the proof of Theorem 2.5. \square

Proof of Theorem 1.1. For each positive integer n , set $m_1(n) = n - n^{3/4}$ and $m_2(n) = n + n^{3/4}$. Recall that the Poisson sequence N_n is assumed to be non decreasing. Let r_n be

as in the statement of the Theorem. It is easy to see that the proof of Theorem 2.5 goes through for $m_i(n)$, that is,

$$W'_{m_i(n)}(r_n) \xrightarrow{\mathcal{D}} Po(e^{-\alpha}/C_d), \quad i = 1, 2. \quad (2.55)$$

Let $\mathcal{P}_n^- = \mathcal{P}_{m_1(n)}$ and $\mathcal{P}_n^+ = \mathcal{P}_{m_2(n)}$. Let A^c denote the complement of set A . Define events H_n, A_n and B_n by

- $H_n = \{\mathcal{P}_n^- \subseteq \mathcal{X}_n \subseteq \mathcal{P}_n^+\}$.
- Let A_n be the event that there exist a point $Y \in \mathcal{P}_n^+ \setminus \mathcal{P}_n^-$ such that Y is isolated in $G(\mathcal{P}_n^- \cup \{Y\}, r_n)$.
- Let B_n be the event that one or more points of $\mathcal{P}_n^+ \setminus \mathcal{P}_n^-$ lies within distance r_n of a point X of \mathcal{P}_n^- with degree zero in $G(\mathcal{P}_n^-, r_n)$.

Then

$$\{W_n(r_n) \neq W'_n(r_n)\} \subseteq A_n \cup B_n \cup F_n^c.$$

The proof is complete if we show that $P(A_n), P(B_n), P(F_n^c)$ all converge to 0.

$$\begin{aligned} P[H_n^c] &\leq P[N_{m_1(n)} \geq n] + P[N_{m_2(n)} \leq n] \\ &\leq P[|N_{m_1(n)} - m_1(n)| \geq n^{3/4}] + P[|N_{m_2(n)} - m_2(n)| \geq n^{3/4}] \rightarrow 0, \end{aligned}$$

by the Chebyshev's inequality.

Let $Y \sim f$ be a point independent of \mathcal{P}_n^- . Evidently,

$$\begin{aligned} P[A_n] &\leq 2n^{3/4} P[Y \text{ is isolated in } G(\mathcal{P}_n^- \cup \{Y\}, r_n)] \\ &= 2n^{3/4} m_1(n)^{-1} E[W'_{m_1(n)}(r_n)] \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

By the Boole's inequality and the Palm theory,

$$\begin{aligned} P[B_n] &\leq 2n^{3/4} P[\text{there is a isolated point of } G(\mathcal{P}_n^-, r_n) \text{ in } B(Y, r_n)] \\ &\leq 2n^{7/4} \int_{\mathbb{R}^d} f(y) dy \cdot \int_{B(y, r_n)} \exp(-m(n)I(x, r_n)) f(x) dx. \end{aligned}$$

By interchanging the order of integration, we obtain

$$\begin{aligned} P(B_n) &\leq 2n^{7/4} \int_{\mathbb{R}^d} I(x, r_n) \exp(-m(n)I(x, r_n)) f(x) dx \\ &= 2n^{3/4} \int_{-a_n}^{\infty} I(\rho_n(t), r_n) \exp(-m(n)I(\rho_n(t), r_n)) g_n(t) dt. \end{aligned} \quad (2.56)$$

From (2.30) and (2.23), we get

$$2n^{3/4} I(\rho_n(t), r_n) \exp(-m(n)I(\rho_n(t), r_n)) g_n(t) \leq Cn^{-1/4} g_n(t) \rightarrow 0.$$

Thus the integrand in (2.56) converges pointwise to 0 as $n \rightarrow \infty$. Proceeding as in the proof of Proposition 2.4, using the integrable bounds obtained in the proof of Proposition 2.4, for $\exp(-m(n)I(\rho_n(t), r_n)) g_n(t)$ and the bounds for $I(\rho_n(t), r_n)$, and the dominated convergence theorem, we get $P[B_n] \rightarrow 0$. This completes the proof. \square

Proof of Theorem 1.2. Let r_n be as in the statement of the Theorem. Then,

$$\lim_{n \rightarrow \infty} P[d_n \leq r_n] = \lim_{n \rightarrow \infty} P[W_n(r_n) = 0] = \exp(-e^{-\beta}/C_d).$$

\square

In order to prove strong law results for the LNND for graphs with densities having compact support, one covers the support of the density using an appropriate collection of concentric balls and then shows summability of certain events involving the distribution of the points of \mathcal{X}_n on these balls. The results then follow by an application of the Borel-Cantelli Lemma.

In case of densities having unbounded support, the region to be covered changes with n and must be determined first. The following Lemma gives us the regions of interest when the points in $\mathcal{X}_n = \{X_1, X_2, \dots, X_n\}$, $n \geq 1$ are distributed according to the probability density function f given by (1.1) .

For any $c \in \mathbb{R}$, and large enough n , define

$$R_n^\alpha(c) = \frac{1}{\lambda} \left(\log n + \frac{c + d - \alpha}{\alpha} \log_2 n \right). \quad (2.57)$$

For any set A , let A^c denote its complement. Let $U_n(c)$ be the event $\mathcal{X}_n \subset B(0, R_n(c))$ and for any $c < 0$, $V_n(c)$ denote the event that at least one point of \mathcal{X}_n lies in $B(0, R_n(0)) \setminus B(0, R_n(c))$. $a_n \gtrsim b_n$ implies that $a_n > c_n$ for some sequence c_n and $c_n \sim b_n$. Further, C, C_1, C_2 , etc., will denote constants whose values might change from place to place.

Lemma 2.6 *Let the events U_n and V_n , $n \geq 1$, be as defined above. Then*

1. $P[U_n^c(c) \text{ i.o.}] = 0$, for any $c > \alpha$, and
2. $P[V_n^c(c) \text{ i.o.}] = 0$, for any $c < 0$.

The above results are also true with \mathcal{X}_n replaced by \mathcal{P}_{λ_n} provided $\lambda_n \sim n$.

Thus for almost all realizations of the sequence $\{\mathcal{X}_n\}_{n \geq 1}$, all points of \mathcal{X}_n will lie within the ball $B(0, R_n(c))$ for any $c > \alpha$ eventually, and for $c < 0$, there will be at least one point of \mathcal{X}_n in $B(0, R_n(0)) \setminus B(0, R_n(c))$ eventually.

Proof of Lemma 2.6. From (1.3), write $f_R(r) = A'_d e^{-\lambda r^\alpha} r^{d-1}$. Note that

$$\int_{\tilde{R}}^{\infty} f_R(r) dr \sim A'_d (\lambda \alpha)^{-1} \tilde{R}^{d-\alpha} e^{-\lambda \tilde{R}^\alpha}, \quad \text{as } \tilde{R} \rightarrow \infty. \quad (2.58)$$

Fix $a > 1$, and define the subsequence $n_k = a^k$. For large k , we have

$$\begin{aligned}
P[\cup_{n=n_k}^{n_{k+1}} U_n^c(c)] &\leq P[\text{at least one vertex of } \mathcal{X}_{n_{k+1}} \text{ is in } B^c(0, R_{n_k}(c))] \\
&= 1 - (I(0, R_{n_k}(c)))^{n_{k+1}} = 1 - (1 - \int_{R_{n_k}(c)}^{\infty} f_R(r) dr)^{n_{k+1}} \\
&\leq n_{k+1} \int_{R_{n_k}(c)}^{\infty} f_R(r) dr \sim A'_d(\lambda\alpha)^{-1} n_{k+1} R_{n_k}^{d-\alpha}(c) e^{-\lambda R_{n_k}^\alpha(c)} \\
&\leq \frac{C}{k^{c/\alpha}}.
\end{aligned}$$

Thus the above probability is summable for $c > \alpha$, and the first part of Lemma 2.6 follows from the Borel-Cantelli Lemma.

Next, let $c < 0$ and take $n_k = a^k$, for some $a > 1$. Note that for all n, m sufficiently large $R_n(c)$ are increasing and $R_n(c) < R_m(0)$. Hence for k sufficiently large, using (2.58) and the inequality $1 - x \leq \exp(-x)$, we get

$$\begin{aligned}
P[\cup_{n=n_k}^{n_{k+1}} V_n^c(c)] &\leq P[\mathcal{X}_{n_k} \cap (B(0, R_{n_k}(0)) \setminus B(0, R_{n_{k+1}}(c))) = \emptyset] \\
&= \left(1 - \int_{R_{n_{k+1}}(c)}^{R_{n_k}(0)} A'_d e^{-\lambda r^\alpha} r^{d-\alpha} dr\right)^{n_k} \\
&\leq \exp\left(-n_k \int_{R_{n_{k+1}}(c)}^{R_{n_k}(0)} A'_d e^{-\lambda r^\alpha} r^{d-\alpha} dr\right) \\
&\leq \exp(-n_k c_1 A'_d(\lambda\alpha)^{-1} (R_{n_{k+1}}^{d-\alpha}(c) e^{-\lambda R_{n_{k+1}}^\alpha(c)} - R_{n_k}^{d-\alpha}(0) e^{-\lambda R_{n_k}^\alpha(0)})) \\
&\leq e^{-c_2 k^{-c/\alpha}}
\end{aligned} \tag{2.59}$$

which is summable for all $c < 0$. The second part of Lemma 2.6 now follows from the Borel-Cantelli Lemma. If \mathcal{X}_n is replaced by \mathcal{P}_{λ_n} , where $\lambda_n \sim n$, then

$$\begin{aligned}
P[U_n^c(c)] &= 1 - \exp(-\lambda_n(1 - I(0, R_n(c)))) \\
&\lesssim \lambda_n A'_d(\lambda\alpha)^{-1} R_n^{d-\alpha}(c) \exp(-\lambda R_n^\alpha(c)) \\
&\sim n A'_d(\lambda\alpha)^{-1} R_n^{d-\alpha}(c) \exp(-\lambda R_n^\alpha(c)),
\end{aligned}$$

which is same as the $P[U_n^c(c)]$ in case of \mathcal{X}_n . Similarly, one can show that $P[V_n^c(c)]$ has the same asymptotic behavior as in the case of \mathcal{X}_n . Thus the results stated for \mathcal{X}_n also hold for \mathcal{P}_{λ_n} .

Proposition 2.7 *Let $t > d/\alpha\lambda$, and let $r_n(t) = t(\lambda^{-1} \log n)^{\frac{1}{\alpha}-1} \log_2 n$. Then with probability 1, $d_n \leq r_n(t)$ for all large enough n .*

Proof. Let $c > \alpha$ and pick u, t such that $(c + \alpha(d-1))/\alpha^2\lambda < u < t$, and $\epsilon > 0$ satisfying

$$\epsilon + u < t.$$

From Lemma 2.6, $\mathcal{X}_n \subset B(0, R_n(c))$ a.s. for all large enough n . For $m = 1, 2, \dots$, let $\nu(m) = a^m$, for some $a > 1$. Let κ_m (the covering number), be the minimum number of balls of radius $r_{\nu(m+1)}(\epsilon)$ required to cover the ball $B(0, R_{\nu(m+1)}(c))$. For large m , we have

$$\begin{aligned} \kappa_m &\leq C_1 \frac{R_{\nu(m+1)}(c)^d}{r_{\nu(m+1)}^d(\epsilon)} \\ &= \frac{(\log(\nu(m+1)) + \frac{c+d-\alpha}{\alpha} \log_2(\nu(m+1)))^{d/\alpha}}{\lambda^d \epsilon^d (\log(\nu(m+1)))^{(d/\alpha-d)} (\log_2(\nu(m+1)))^d} \\ &\leq C_2 \left(\frac{m+1}{\log(m+1)} \right)^d. \end{aligned} \tag{2.60}$$

Consider the deterministic set $\{x_1^m, \dots, x_{\kappa_m}^m\} \subset B(0, R_{\nu(m+1)}(c))$, such that

$$B(0, R_{\nu(m+1)}(c)) \subset \cup_{i=1}^{\kappa_m} B(x_i^m, r_{\nu(m+1)}(\epsilon)).$$

Let $\alpha > 1$. Given $x \in \mathbb{R}^d$, define $A_m(x)$ to be the annulus $B(x, r_{\nu(m+1)}(u)) \setminus B(x, r_{\nu(m+1)}(\epsilon))$, and let $F_m(x)$ be the event such that no vertex of $\mathcal{X}_{\nu(m)}$ lies in $A_m(x)$, i.e.

$$F_m(x) = \{\mathcal{X}_{\nu(m)}[A_m(x)] = 0\}, \tag{2.61}$$

where $\mathcal{X}[B]$ denotes the number of points of the finite set \mathcal{X} that lie in B . For any $x \in$

$B(0, R_{\nu(m+1)}(c))$, we have

$$\begin{aligned} P[X_i \in A_m(x)] &= \int_{A_m(x)} f(y) dy \\ &\geq \int_{A_m(R_{\nu(m+1)}(c)e)} f(y) dy \\ &= I(R_{\nu(m+1)}(c), r_{\nu(m+1)}(u)) - I(R_{\nu(m+1)}(c), r_{\nu(m+1)}(\epsilon)). \end{aligned}$$

Since $R_n(c), r_n$ satisfy the conditions of Lemma 2.1, we have for large m ,

$$\begin{aligned} P[X_i \in A_m(x)] &\geq e^{-\lambda R_{\nu(m+1)}^\alpha(c)} (R_{\nu(m+1)}^{\alpha-1}(c))^{-\frac{d+1}{2}} \\ &\quad \cdot \left(c_1 e^{\lambda \alpha r_{\nu(m+1)}(u) R_{\nu(m+1)}^{\alpha-1}(c)} (r_{\nu(m+1)}(u))^{\frac{d-1}{2}} - c_2 e^{\lambda \alpha r_{\nu(m+1)}(\epsilon) R_{\nu(m+1)}^{\alpha-1}(c)} (r_{\nu(m+1)}(\epsilon))^{\frac{d-1}{2}} \right) \\ &:= q_m. \end{aligned}$$

Substituting the values of $R_{\nu(m+1)}(c)$ and $r_{\nu(m+1)}(\cdot)$ in q_m , we get for large m

$$q_m \leq (C(u) - C(\epsilon)) \frac{(\log(m+1))^{(d-1)/2}}{a^{m+1} (m+1)^{c/\alpha + d - \alpha \lambda u - 1}}. \quad (2.62)$$

Hence, for large m , we have

$$P[F_m(x)] \leq (1 - q_m)^{\nu(m)} \leq \exp(-\nu(m)q_m) \leq \exp\left(-C \frac{(\log(m+1))^{(d-1)/2}}{m^{\frac{c}{\alpha} + d - \alpha \lambda u - 1}}\right). \quad (2.63)$$

Set $G_m = \cup_{i=1}^{\kappa_m} F_m(x_i^m)$. From (2.60) and (2.63), we have for large m ,

$$\begin{aligned} P[G_m] &= P[\cup_{i=1}^{\kappa_m} F_m(x_i^m)] \leq \sum_{i=1}^{\kappa_m} P[F_m(x_i^m)] \\ &\leq C_2 \left(\frac{m+1}{\log(m+1)} \right)^d \exp\left(-C \frac{(\log(m+1))^{(d-1)/2}}{(m+1)^{c/\alpha + d - \alpha \lambda u - 1}}\right), \end{aligned} \quad (2.64)$$

which is summable in m since $u > \frac{c + \alpha(d-1)}{\alpha^2 \lambda}$. By Borel-Cantelli Lemma, G_m occurs only for

finitely many m a.s.

Pick n , and take m such that $a^m \leq n \leq a^{m+1}$. If $d_n \geq r_n(t)$, then there exists an $X \in \mathcal{X}_n$ such that $\mathcal{X}_n[B(X, r_n(t)) \setminus \{X\}] = 0$. By Lemma 2.6, X will be in $B(0, R_{\nu(m+1)}(c))$ for all large enough n , so there is some $i \leq \kappa_m$ such that $X \in B(x_i^m, r_{\nu(m+1)}(\epsilon))$. So, if m is large enough,

$$r_{\nu(m+1)}(\epsilon) + r_{\nu(m+1)}(u) \leq r_{\nu(m+1)}(t) \leq r_n(t).$$

So, $F_m(x_i)$ and hence G_m occur. since G_m occurs finitely often a.s., $d_n \leq r_n(t)$ for all large n , a.s. The result now follows since $c > \alpha$ is arbitrary.

In the case when $\alpha \leq 1$, cover the ball $B(0, R_{\nu(m+1)}(c_1))$, by the balls of radius $r_{\nu(m)}(\epsilon)$ and define the annulus $A_m(x)$ to be $B(x, r_{\nu(m)}(u)) \setminus B(x, r_{\nu(m)}(\epsilon))$. Take $F_m(x) = \{\mathcal{X}_{\nu(m+1)}[A_m(x)] = 0\}$ and proceed as in the case $\alpha > 1$. This completes the proof of Proposition 2.7. \square

Now we derive a lower bound for d_n . Let $r_n(t) = t \log_2 n (\lambda^{-1} \log n)^{1/\alpha-1}$.

Proposition 2.8 *Let $t < (d-1)/\alpha\lambda$. Then with probability 1, $d_n \geq r_n(t)$, eventually.*

Proof. We prove the above proposition using the Poissonization technique, which uses the following Lemma (see Lemma 1.4, Penrose [8]).

Lemma 2.9 *Let $N(\lambda)$ be Poisson a random variable with mean λ . Then there exists a constant c such that for all $\lambda > \lambda_1$,*

$$P[X > \lambda + \lambda^{3/4}/2] \leq c \exp(-\lambda^{1/2}),$$

and

$$P[X < \lambda - \lambda^{3/4}/2] \leq c \exp(-\lambda^{1/2}).$$

Enlarging the probability space, assume that for each n there exist Poisson variables $N(n)$ and $M(n)$ with means $n - n^{3/4}$ and $2n^{3/4}$ respectively, independent of each other and of $\{X_1, X_2, \dots\}$. Define the point processes

$$\mathcal{P}_n^- = \{X_1, X_2, \dots, X_{N(n)}\}, \quad \mathcal{P}_n^+ = \{X_1, X_2, \dots, X_{N(n)+M(n)}\}.$$

Then, \mathcal{P}_n^- and \mathcal{P}_n^+ are Poisson point processes on \mathbb{R}^d with intensity functions $(n - n^{3/4})f(\cdot)$ and $(n + n^{3/4})f(\cdot)$ respectively. The point processes \mathcal{P}_n^- , \mathcal{P}_n^+ and \mathcal{X}_n are coupled in such a way that $\mathcal{P}_n^- \subset \mathcal{P}_n^+$. Thus, if $H_n = \{\mathcal{P}_n^- \subset \mathcal{X}_n \subset \mathcal{P}_n^+\}$, then by the Borel-Cantelli Lemma and Lemma 2.9, $P[H_n^c \text{ i.o.}] = 0$. Hence $\{\mathcal{P}_n^- \subset \mathcal{X}_n \subset \mathcal{P}_n^+\}$ a.s. for all large enough n .

Pick constants u, c, t, ϵ such that $c < 0$, $\epsilon > 0$, $0 < t < u < (c + \alpha(d-1))/\alpha^2\lambda$, and $\epsilon + t < u$.

Consider the annulus $A_n(c) = B(0, R_n(0)) \setminus B(0, R_n(c))$, $c < 0$, where $R_n(c)$ is as defined in (2.57). For each n , choose a non-random set $\{x_1^n, x_2^n, \dots, x_{\sigma_n}^n\} \subset A_n(c)$, such that the balls $B(x_i^n, r_n(u))$, $1 \leq i \leq \sigma_n$ are disjoint. The packing number σ_n is the maximum number of disjoint balls $B(x, r_n(u))$, with $x \in A_n(c)$. For large n , we have

$$\begin{aligned} \sigma_n &\geq c_1 \frac{R_n^d(0) - R_n^d(c)}{r_n^d(u)} \\ &= \frac{(\log n + \frac{d-\alpha}{\alpha} \log_2 n)^{d/\alpha} - (\log n + \frac{c+d-\alpha}{\alpha} \log_2 n)^{d/\alpha}}{\lambda^{d/\alpha} r_n^d(u)} \\ &\geq c_2 \left(\frac{\log n}{\log_2 n} \right)^{d-1}. \end{aligned} \tag{2.65}$$

By Lemma 2.6, there will be points in A_n for all large enough n , a.s. Fix $a > 1$ and let $\nu(k) = a^k$, $k = 0, 1, 2, \dots$. Consider the sequence of sets

$$\left(\bigcup_{i=1}^{\sigma_{\nu(m)}} E_{m,i} \right)^c, \tag{2.66}$$

where,

$$E_{m,i} = \{\mathcal{P}_{\nu(m)}^- [B(x_i^{\nu(m)}, r_{\nu(m)}(\epsilon))] = 1\} \cap \{\mathcal{P}_{\nu(m+1)}^+ [B(x_i^{\nu(m)}, r_{\nu(m)}(u))] = 1\},$$

where $i = 1, 2, \dots, \sigma_{\nu(m)}$, $m = 1, 2, \dots$. From an earlier argument $P[H_n^c]$ is summable and hence H_n happens eventually w.p.1. For any n , let m be such that $\nu(m) \leq n \leq \nu(m+1)$. If H_n and $E_{m,i}$ happen, then there is a point of $X \in \mathcal{P}_n^- \subset \mathcal{X}_n$ such that $X \in B(x_i^{\nu(m)}, r_{\nu(m)}(\epsilon))$ with no other point of \mathcal{P}_n^+ (and hence of \mathcal{X}_n) in $B(x_i^{\nu(m)}, r_{\nu(m)}(u))$. This would imply that $d_n \geq r_{\nu(m)}(t) \geq r_n(t)$. Thus the proof is complete if we show that

$$\sum_{m=1}^{\infty} P \left[\left(\bigcup_{i=1}^{\sigma_{\nu(m)}} E_{m,i} \right)^c \right] < \infty,$$

To this end, we first estimate $P[E_{m,i}]$.

Set $\mathcal{I}_m = \mathcal{P}_{\nu(m+1)}^+ \setminus \mathcal{P}_{\nu(m)}^-$, and let $U_{m,i} = B(x_i^{\nu(m)}, r_{\nu(m)}(\epsilon))$, and $V_{m,i} = B(x_i^{\nu(m)}, r_{\nu(m)}(u)) \setminus U_{m,i}$. Then,

$$E_{m,i} = \{\mathcal{P}_{\nu(m)}^- [U_{m,i} = 1]\} \cap \{\mathcal{P}_{\nu(m)}^- [V_{m,i} = 0]\} \cap \{\mathcal{I}_m [U_{m,i} = 0]\} \cap \{\mathcal{I}_m [V_{m,i} = 0]\}.$$

Let $\alpha(m) = \nu(m) - \nu(m)^{3/4}$ and $\beta(m) = (\nu(m+1)) + (\nu(m+1))^{3/4}$. Note that each of the four events appearing in the above equation are independent and that $\alpha(m) \sim \nu(m)$ and $\beta(m) \sim a\nu(m)$. Using this and the Lemma 2.1, we get, for all large enough m ,

$$\begin{aligned} P[E_{m,i}] &= \alpha(m) \int_{U_{m,i}} f(y) dy \exp \left(-\alpha(m) \int_{U_{m,i}} f(y) dy \right) \exp \left(-\alpha(m) \int_{V_{m,i}} f(y) dy \right) \\ &\quad \exp \left(-(\beta(m) - \alpha(m)) \int_{U_{m,i}} f(y) dy \right) \exp \left(-(\beta(m) - \alpha(m)) \int_{V_{m,i}} f(y) dy \right) \\ &= \alpha(m) \int_{U_{m,i}} f(y) dy \exp \left(-\beta(m) \int_{U_{m,i} \cup V_{m,i}} f(y) dy \right) \end{aligned}$$

$$\begin{aligned}
&= \alpha(m) I(x_i^{\nu(m)}, r_{\nu(m)}(\epsilon)) \exp\left(-\beta(m) I(x_i^{\nu(m)}, r_{\nu(m)}(u))\right) \\
&\geq \alpha(m) I(R_{\nu(m)}(0), r_{\nu(m)}(\epsilon)) \exp\left(-\beta(m) I(R_{\nu(m)}(c), r_{\nu(m)}(u))\right) \\
&\geq C_1 \nu(m) r_{\nu(m)}^d(\epsilon) e^{-\lambda(R_{\nu(m)}^\alpha(0) - \alpha r_{\nu(m)}(\epsilon) R_{\nu(m)}^{\alpha-1}(0))} (\lambda \alpha r_{\nu(m)}(\epsilon) R_{\nu(m)}^{\alpha-1}(0))^{-\frac{d+1}{2}} \\
&\quad \cdot \exp\left(-C_2 \nu(m) r_{\nu(m)}(u)^d e^{-\lambda(R_{\nu(m)}^\alpha(c) - \alpha r_{\nu(m)}(\epsilon) R_{\nu(m)}^{\alpha-1}(c))} (\lambda \alpha r_{\nu(m)}(\epsilon) R_{\nu(m)}^{\alpha-1}(c))^{-\frac{d+1}{2}}\right) \\
&\sim C_3 \frac{(\log_2(\nu(m)))^{\frac{d-1}{2}}}{(\log(\nu(m)))^{d-1-\alpha\epsilon\lambda}} \exp\left(-C_4 \frac{(\log_2(\nu(m)))^{\frac{d-1}{2}}}{(\log(\nu(m)))^{d+c/\alpha-\alpha u\lambda-1}}\right) \\
&\sim C_3 \frac{(\log_2(\nu(m)))^{\frac{d-1}{2}}}{(\log(\nu(m)))^{d-1-\alpha\epsilon\lambda}}, \tag{2.67}
\end{aligned}$$

where the last relation follows since $u < \frac{\alpha d + c - \alpha}{\alpha^2 \lambda}$. The events $E_n(x_i^n)$, $1 \leq i \leq \sigma_n$ are independent, so by (2.67), for large enough m ,

$$\begin{aligned}
P\left[\left(\bigcup_{i=1}^{\sigma_{\nu(m)}} E_{m,i}\right)^c\right] &\leq \prod_{i=1}^{\sigma_{\nu(m)}} \exp(-P[E_{m,i}]) \\
&\leq \exp\left(-C_5 \sigma_{\nu(m)} \frac{(\log_2(\nu(m)))^{\frac{d-1}{2}}}{(\log(\nu(m)))^{d-1-\alpha\epsilon\lambda}}\right) \\
&\leq \exp\left(-C_6 \left(\frac{m}{\log m + \log_2 a}\right)^{d-1} \frac{(\log m + \log_2 a)^{\frac{d-1}{2}}}{m^{d-1-\alpha\epsilon\lambda}}\right) \\
&= \exp\left(-C_6 \frac{m^{\alpha\epsilon\lambda}}{(\log m + \log_2 a)^{(d-1)/2}}\right),
\end{aligned}$$

which is summable in m .

In case when $\alpha \leq 1$. Define $U_{m,i} = B(x_i^{\nu(m)}, r_{\nu(m+1)}(\epsilon))$, and $V_{m,i} = B(x_i^{\nu(m)}, r_{\nu(m+1)}(u)) \setminus U_{m,i}$. Proceeding as above we can show that $P\left[\left(\bigcup_{i=1}^{\sigma_{\nu(m)}} E_{m,i}\right)^c\right]$ is summable. This gives $d_n \geq r_{\nu(m+1)}(t) \geq r_n(t)$. This completes the proof of Proposition 2.8. \square

Proof of Theorem 1.3. Immediate from Proposition 2.7 and Proposition 2.8.

References

- [1] Appel, M.J.B. and Russo, R.P. (1997), The minimum vertex degree of a graph on the uniform points in $[0, 1]^d$, *Advances in Applied Probability*, **29**, 582-594.
- [2] Gupta, B., Iyer, S.K. and Manjunath, D. (2005), On the Topological Properties of One Dimensional Exponential Random Geometric Graphs, to appear.
- [3] Henze, N. and Dette, H. (1989), The limit distribution of the largest nearest-neighbor link in the unit d-cube, *Journal of Applied Probability*, **26(1)**, 67-80.
- [4] Hsing, T. and Rootzen, H. (2005), Extremes on trees, *Annals of Probability*, **33(1)**, 413 - 444.
- [5] Penrose, M. (1997), The longest edge of the minimal spanning tree, *Annals of Applied Probability*, **7**, 340-361.
- [6] Penrose, M. (1998), Extremes for the minimal spanning tree on the Normally distributed points, *Advances in Applied Probability*, **30**, 628-639.
- [7] Penrose, M. (1999), A strong law for the largest nearest neighbor link between random points, *Journal of the london mathematical society*, **60**, 951-960.
- [8] Penrose, M.(2003), Random Geometric Graphs, second edition, *Oxford University Press*.
- [9] Steele, J. M. and Tierney, L. (1986), Boundary domination and the distribution of the largest nearest-neighbor link, *Journal of Applied Probability*, **23**, 524-528.